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Dynamic effects on the stretching of the magnetic field by a plasma flow

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Abstract

A key mechanism in the growth of magnetic energy in kinematic dynamos is the stretching of the magnetic field vector by making it point in an unstable direction of the strain matrix. Our objective is to study whether this feature may be maintained in an ideal plasma when also considering the back reaction of the magnetic field upon the flow through the Lorentz force. Several effects occur: in addition to the nonlocal ones exerted by the total pressure, a complex geometry of magnetic field lines decreases the rate of growth of magnetic energy, rotation of the flow enhances it and above all the rate of growth decreases with minus the square of the eigenvalue associated with the magnetic field direction. Thus local dynamics tend to rapidly quench the stretching of the field.

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1. Introduction

The physics of magnetic field growth in magnetohydrodynamic plasmas is well understood at the kinematic level. That is, if we assume that the (incompressible) plasma flow is not influenced by the magnetic field, the evolution of this is governed only by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} - \mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} \tag{1}$$

where $\mathbf{u}$ represents the flow velocity, $\mathbf{B}$ represents the magnetic field and η represents the resistivity or magnetic diffusivity, usually very small in the astrophysical problems to which this theory is mostly applied. If we omit the diffusion (ideal plasmas), (1) may be written in terms of the Lagrangian derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

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as

$$\frac{D\mathbf{B}}{Dt} = \mathbf{B}\cdot\nabla\mathbf{u}. \tag{2}$$

From this it follows

$$\frac{1}{2}\frac{DB^2}{Dt} = \mathbf{B}\cdot\nabla\mathbf{u}(\mathbf{B}) = \mathbf{B}\cdot S(\mathbf{B}) \tag{3}$$

where $S = (1/2)(\nabla\mathbf{u} + (\nabla\mathbf{u})^{+})$ is the strain matrix $(1/2)(\partial_i u_j + \partial_j u_i)$ and $(\cdots)^{+}$ represents the transposed matrix. Following the Einstein convention of summation on repeated indices, the vector $\nabla\mathbf{u}(\mathbf{B})$ is $(\partial_i u_j)B_i$, the same as the directional derivative $\mathbf{B}\cdot\nabla\mathbf{u}$. S possesses an orthogonal system of eigenvectors: its three (real) eigenvalues add to zero. Obviously, the largest growth of magnetic energy along a streamline occurs when $\mathbf{B}$ aligns with the eigenvector of largest eigenvalue, i.e. the most unstable direction of the flow. By integrating (3) in a domain Ω such that $\mathbf{u}\cdot\mathbf{n}\mid_{\partial\Omega}= 0$, $\mathbf{B}\cdot\mathbf{n}\mid_{\partial\Omega}= 0$ (or in a periodic box), we obtain the classical relation on the magnetic energy:

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega B^2\,\mathrm{d}V = \int_\Omega \mathbf{B}\cdot S(\mathbf{B})\,\mathrm{d}V. \tag{4}$$

Most models of kinematic dynamos tend to align $\mathbf{B}$ with the most unstable eigenvector of S. To have a global exponential growth, we need the largest eigenvalue of S to be positive at least in a set of positive measure in Ω. Since the transport of vectors by the flow also satisfies (2) (which is the reason why magnetic field lines are carried as material points), this implies exponential stretching of the flow, which is an indication of chaos. That the flow must be chaotic for an exponentially growing magnetic field was proved rigorously in [1]. The problem is that such flows tend to produce a magnetic field pointing in opposite directions in contiguous sheets or ropes of the domain [2], so that when adding the diffusive term, dissipation of the energy is intense. Thus kinematic dynamos also need constructive folding: the flow must be such that the contribution of the field at nearby sheets does not cancel out. This is indeed the main hurdle in constructing models of fast dynamos (see [3] and references therein).

While kinematic dynamos provide valuable insights, realistic dynamos must take into account the full MHD system. This complex problem can only be studied analytically by concentrating on particular aspects. We will consider for how long we can maintain a magnetic field pointing approximately in an unstable direction of the flow, so that there is exponential increase of the magnetic energy. There are reasons to feel optimistic about this, since the vorticity in the Navier–Stokes equations also satisfies the induction equation and has been found to have a tendency to align with the intermediate vector of the strain matrix. This was pointed first in [4], confirmed numerically [5–8] and given a tentative kinematic explanation [9, 10]. Some of these simulations seem to show that the vorticity points first to the largest eigenvector, and as the turbulence develops, it points to the second largest one [4, 5, 8]. An explanation of this phenomenon in terms of attracting points is given in [11]. However, it is dangerous to extrapolate this to the magnetic field, since the Lorentz force imposes a forcing absent in the vorticity case: indeed, we will find a different phenomenology.

Since we are dealing with a purely dynamic problem, we will ignore the actions of both viscosity and resistivity and therefore deal with ideal plasmas. For these the MHD equations may be written as

$$\frac{D\mathbf{u}}{Dt} = \mathbf{B}\cdot\nabla\mathbf{B} - \nabla p_* + \mathbf{g} \tag{5}$$

$$\frac{D\mathbf{B}}{Dt} = \mathbf{B}\cdot\nabla\mathbf{u} \tag{6}$$

where $p_* = p + (1/2)B^2$ is the total (kinetic plus magnetic) pressure, and $\mathbf{g}$ is a possible forcing upon the momentum equation: forcings on the induction one are infrequent. We will study the evolution of the rate of increase of the magnetic energy $\Phi = \mathbf{B} \cdot S(\mathbf{B})$, and in order to consider more specifically the alignment of the field with the eigenvectors of S, the normalized value $\Psi = \Phi/B^2 = \mathbf{b} \cdot S(\mathbf{b})$, where $\mathbf{b} = \mathbf{B}/B$ is the unit magnetic field. Care must be taken with the null points of $\mathbf{B}$, where Ψ ceases to be defined.

2. Evolution of the magnetic energy variation

As stated, let us define $\Phi = \mathbf{B} \cdot S(\mathbf{B})$. We will study the Lagrangian derivative of Φ. We have

$$\begin{aligned}\frac{D\Phi}{Dt} = \frac{DB_i}{Dt} B_j(\partial_i u_j + \partial_j u_i) + \frac{1}{2} B_i B_j \left(\frac{D}{Dt}(\partial_i u_j + \partial_j u_i) \right) \\ = B_k(\partial_k u_i) B_j(\partial_i u_j + \partial_j u_i) + \frac{1}{2} B_i B_j \left(\partial_i \frac{\partial u_j}{\partial t} + \partial_j \frac{\partial u_i}{\partial t} \right) \\ + \frac{1}{2} B_i B_j (u_k \partial_{k,i} u_j + u_k \partial_{k,j} u_i). \qquad (7)\end{aligned}$$

By using the induction equation on B_i and the momentum equation on $\partial u_i/\partial t$, we obtain

$$\begin{aligned}\frac{D\Phi}{Dt} = B_j B_k[(\partial_k u_i)(\partial_i u_j) + (\partial_k u_i)(\partial_j u_i)] + \frac{1}{2} B_i B_j (u_k \partial_{k,i} u_j + u_k \partial_{k,j} u_i) \\ + \frac{1}{2} B_i B_j \left[\partial_i \left(-u_k \partial_k u_j + B_k \partial_k B_j - \frac{1}{2} \partial_j B^2 - \partial_j p + g_j \right) \right. \\ \left. + \partial_j \left(-u_k \partial_k u_i + B_k \partial_k B_i - \frac{1}{2} \partial_i B^2 - \partial_i p + g_i \right) \right]. \qquad (8)\end{aligned}$$

After making the derivation in the third term and performing some cancellations, we find

$$\begin{aligned}\frac{D\Phi}{Dt} = B_i B_j (\partial_i u_j)(\partial_j u_k) + \frac{1}{2} B_i B_j [(\partial_i B_k)(\partial_k B_j) \\ + (\partial_j B_k)(\partial_k B_i) - 2(\partial_i B_k)(\partial_j B_k) - 2\partial_{i,j} p + \partial_i g_j + \partial_j g_i] \qquad (9)\end{aligned}$$

which, written in vectorial form, yields

$$\frac{D\Phi}{Dt} = |\nabla \mathbf{u}(\mathbf{B})|^2 - |\nabla \mathbf{B}(\mathbf{B})|^2 + \mathbf{B} \cdot (\nabla \mathbf{B})^2(\mathbf{B}) - \mathbf{B} \cdot P''(\mathbf{B}) + \mathbf{B} \cdot \frac{1}{2}(\nabla \mathbf{g} + (\nabla \mathbf{g})^+)(\mathbf{B}) \qquad (10)$$

where by P'' we understand the Hessian matrix of p: $\mathbf{B} \cdot P''(\mathbf{B}) = B_i B_j \partial_{i,j} p$. Since

$$\mathbf{B} \cdot (\nabla \mathbf{B})^2(\mathbf{B}) = \tfrac{1}{2} \mathbf{B} \cdot \nabla(\mathbf{B} \cdot \nabla B^2) - \tfrac{1}{2} \mathbf{B} \cdot (B^2)''(\mathbf{B}) \qquad (11)$$

and taking into account the definition of p_*, denoting $G = (1/2)(\nabla \mathbf{g} + (\nabla \mathbf{g})^+)$, the previous equation may be written as

$$\frac{D\Phi}{Dt} = |\nabla \mathbf{u}(\mathbf{B})|^2 - |\nabla \mathbf{B}(\mathbf{B})|^2 + \frac{1}{2} \mathbf{B} \cdot \nabla(\mathbf{B} \cdot \nabla B^2) - \mathbf{B} \cdot P_*''(\mathbf{B}) + \mathbf{B} \cdot G(\mathbf{B}). \qquad (12)$$

Notice that the integral of the term $\mathbf{B} \cdot \nabla(\mathbf{B} \cdot \nabla B^2)$ in Ω vanishes, so that the net contribution of this term is zero. Since the integral of $\mathbf{u} \cdot \nabla \Phi$ is also zero, (12) may be used to find the second variation of the magnetic energy in Ω:

$$\frac{1}{2} \frac{\partial^2}{\partial t^2} \int_\Omega B^2 \, \mathrm{d}V = \int_\Omega |\nabla \mathbf{u}(\mathbf{B})|^2 - |\nabla \mathbf{B}(\mathbf{B})|^2 - \mathbf{B} \cdot P_*''(\mathbf{B}) + \mathbf{B} \cdot G(\mathbf{B}) \, \mathrm{d}V. \qquad (13)$$

All the terms in (12) act locally, i.e. depend only on the values of $\mathbf{u}$, $\mathbf{B}$ and $\mathbf{g}$ in a neighbourhood of the point under study except for the total pressure. This is the solution of the elliptic equation obtained by taking the divergence in (5):

$$\Delta p_* = \partial_i B_j \partial_j B_i - \partial_i u_j \partial_j u_i + \nabla \cdot \mathbf{g}. \tag{14}$$

It is convenient to transform the independent term as a function of the strain matrix $S = (1/2)(\partial_i u_j + \partial_j u_i)$, the magnetic strain matrix $T = (1/2)(\partial_i B_j + \partial_j B_i)$, the vorticity of the flow ω and the current density $\mathbf{J}$. Since

$$\Delta p_* = \mathrm{Tr}((\nabla \mathbf{B})^2) - \mathrm{Tr}((\nabla \mathbf{u})^2) + \nabla \cdot \mathbf{g} \tag{15}$$

where Tr denotes the trace, let us take the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ formed by the eigenvectors of S. Then

$$(\nabla \mathbf{u})(\mathbf{e}_i) = S\mathbf{e}_i + \tfrac{1}{2}\omega \times \mathbf{e}_i = \lambda_i \mathbf{e}_i + \tfrac{1}{2}\omega \times \mathbf{e}_i \tag{16}$$

and therefore

$$(\nabla \mathbf{u})^2(\mathbf{e}_i) = \lambda_i^2 \mathbf{e}_i + S\left(\tfrac{1}{2}\omega \times \mathbf{e}_i\right) + \tfrac{1}{2}\omega \times \lambda_i \mathbf{e}_i + \tfrac{1}{4}\omega \times (\omega \times \mathbf{e}_i). \tag{17}$$

The second and third terms are orthogonal to $\mathbf{e}_i$. As for the fourth, since

$$[\omega \times (\omega \times \mathbf{e}_i)] \cdot \mathbf{e}_i = -|\omega \times \mathbf{e}_i|^2$$

we obtain

$$\mathbf{e}_i \cdot (\nabla \mathbf{u})^2(\mathbf{e}_i) = \lambda_i^2 - \tfrac{1}{4}|\omega \times \mathbf{e}_i|^2.$$

The sum of the squares of the three eigenvalues is the square of the norm $\|S\|_2$, given by

$$\|S\|_2^2 = \sum_{i,j} |\partial_i u_j + \partial_j u_i|^2.$$

On the other hand, $|\omega \times \mathbf{e}_i|^2$ is the sum of the squares of the components of ω in the remaining vectors $\mathbf{e}_j, \mathbf{e}_k$; by adding all of them we find

$$\mathrm{Tr}(\nabla \mathbf{u})^2 = \|S\|_2^2 - \tfrac{1}{2}|\omega|^2. \tag{18}$$

By performing the same calculation with the eigenvectors of T, we find the equation satisfied by p_*:

$$\Delta p_* = \|T\|_2^2 - \tfrac{1}{2}|\mathbf{J}|^2 - \|S\|_2^2 + \tfrac{1}{2}|\omega|^2 + \nabla \cdot \mathbf{g} \tag{19}$$

which will be useful in determining the possible effect of the pressure term on Φ.

The term occurring in (12) has the form $\mathbf{B} \cdot P_*''(\mathbf{B})$. Although we know the sum of the three second derivatives of P_*'' in any three orthonormal directions, we cannot know *a priori* the contribution of each of them, and in particular the term above. This is obvious even in the simplest example, harmonic functions: the behaviour in a particular direction depends upon the values of the function at the whole boundary, and it cannot be determined locally. In our case we should pose a whole boundary value or transmission problem, find the appropriate Green function and express $\mathbf{B} \cdot P_*''(\mathbf{B})$ as an integral. This is clearly impracticable, and therefore this nonlocal term is the hardest to analyse. One thing we know for sure, which is that it cannot be neglected: since the sum of the second derivatives obtained in (19) is apparently of the same order as the first local terms occurring in (12), at least in some direction its effect is important enough to modify the dynamics. We can see this more clearly by making a naive assumption: if we are in the early stages of a dynamo, so that both $\mathbf{B}$ and $\nabla \mathbf{B}$ are of lower order than $\nabla \mathbf{u}$, we may omit the quartic terms on the field in (12). In the absence of forcing, and omitting the pressure term, we find

$$\frac{D\Phi}{Dt} \sim |\nabla \mathbf{u}(\mathbf{B})|^2. \tag{20}$$

Since $|\Phi| = |\mathbf{B}\cdot\nabla\mathbf{u}(\mathbf{B})| \leqslant |\mathbf{B}||\nabla\mathbf{u}(\mathbf{B})|$, we obtain approximately

$$\frac{D\Phi}{Dt} \geqslant |\mathbf{B}|^{-2}\Phi^2 \tag{21}$$

which actually predicts a blow-up of Φ, as soon as the integral along the streamline of $|\mathbf{B}|^{-2}$ equals $\Phi(0)^{-1}$, which will happen rather quickly given that $|\mathbf{B}|$ is small. This is physically absurd, and emphasizes the important role of the pressure in damping local instabilities. Since in the next section we will obtain a result stating that exponential growth of the field is itself damped by the local terms, we may expect that the effect of the pressure is not so important in this situation.

3. The local dynamics of magnetic field stretching

In ideal MHD, streamlines such that $\mathbf{B}$ does not vanish at any point have the same property through time. Let us take one of these and define the unit magnetic field as $\mathbf{b} = \mathbf{B}/B$. The magnitude

$$\Psi = \Phi/B^2 = \mathbf{B}\cdot S(\mathbf{b}) \tag{22}$$

does not represent the rate of increase of magnetic energy as before, but the efficiency of the magnetic field direction to increase magnetic energy by pointing in an appropriate direction. As stated in the introduction, this is maximal when $\mathbf{b}$ is the unit vector of the strain matrix with largest eigenvalue. We will study whether the MHD system allows this situation to continue for long. Since obviously

$$\frac{D\Psi}{Dt} = \frac{1}{B^2}\frac{D\Phi}{Dt} - \Phi\frac{1}{B^4}\frac{DB^2}{Dt} \tag{23}$$

and

$$\frac{DB^2}{Dt} = 2\mathbf{B}\cdot\frac{D\mathbf{B}}{Dt} = 2\Phi \tag{24}$$

we have

$$\frac{D\Psi}{Dt} = \frac{1}{B^2}\frac{D\Phi}{Dt} - 2\Psi^2. \tag{25}$$

On the other hand, from (12) it follows

$$\frac{1}{B^2}\frac{D\Phi}{Dt} = |\nabla\mathbf{u}(\mathbf{b})|^2 - |\nabla\mathbf{B}(\mathbf{b})|^2 + \frac{1}{2B^2}\mathbf{B}\cdot\nabla(\mathbf{B}\cdot\nabla B^2) - \mathbf{b}\cdot P_*''(\mathbf{b}) + \mathbf{b}\cdot G(\mathbf{b}). \tag{26}$$

Let us analyse the third term. We have

$$\begin{aligned}\frac{1}{2B^2}B_i\partial_i(B_j\partial_j B^2) &= \frac{1}{2}B_i\partial_i\left(\frac{1}{B^2}B_j\partial_j B^2\right) - \frac{1}{2}B_iB_j(\partial_j B^2)\partial_i\left(\frac{1}{B^2}\right)\\ &= \frac{1}{2}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla B^2}{B^2}\right) + \frac{1}{2B^4}(B_iB_j(\partial_i B^2)(\partial_j B^2))\\ &= \frac{1}{2}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla B^2}{B^2}\right) + \frac{1}{2B^2}(\mathbf{b}\cdot\nabla B^2)^2.\end{aligned} \tag{27}$$

Since

$$\frac{\mathbf{b}\cdot\nabla B^2}{B} = 2\mathbf{b}\cdot\nabla\mathbf{B}(\mathbf{b}) = 2\mathbf{b}\cdot T(\mathbf{b}) \tag{28}$$

where we recall that T is the magnetic strain matrix; we find a rather symmetric form for the equation describing the evolution of Ψ:

$$\frac{D\Psi}{Dt} = -2(\mathbf{b}\cdot S(\mathbf{b}))^2 + |\nabla\mathbf{u}(\mathbf{b})|^2 + 2(\mathbf{b}\cdot T(\mathbf{b}))^2 - |\nabla\mathbf{B}(\mathbf{b})|^2 - \mathbf{b}\cdot P''_*(\mathbf{b}) + \mathbf{b}\cdot G(\mathbf{b}) + \frac{1}{2}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla B^2}{B^2}\right). \tag{29}$$

The only discordant term is $F = (1/2)\mathbf{B}\cdot\nabla(\mathbf{B}\cdot\nabla B^2/B^2)$. Any term of the form $\mathbf{B}\cdot\nabla\phi$ integrates to zero over Ω, so we could dismiss this term by saying that its net contribution is zero. This is true (although it does not prevent F from having local important effects), provided ϕ is smooth throughout Ω, whereas in our case there may exist points where the magnetic field vanishes and the definition of ϕ becomes uncertain. However, ϕ remains bounded as long as the size B of the magnetic field has directional derivatives at every direction, which certainly happens in all the classical critical points. Although B is not usually differentiable, B decreases at least like the distance to the critical point: nonalgebraic zeroes of B are quite unusual. The critical set is in fact often formed by isolated points, or smooth curves or surfaces. If, as it happens in these instances, the critical set is bounded by smooth surfaces approaching it while keeping their area bounded, the use of Gauss' theorem and standard approximation procedures will convince us that indeed the mean of F is zero. Only an extremely complicated critical set could change matter.

Let us return to (29). By the standard decomposition of a matrix into its symmetric and antisymmetric parts, we get

$$|\nabla\mathbf{u}(\mathbf{b})|^2 = |S(\mathbf{b}) + \tfrac{1}{2}\omega\times\mathbf{b}|^2 = |S(\mathbf{b})|^2 + \tfrac{1}{4}|\omega\times\mathbf{b}|^2 + S(\mathbf{b})\cdot(\omega\times\mathbf{b}) \tag{30}$$

where as before ω represents the flow vorticity. Doing the same with the current density $\mathbf{J}$,

$$|\nabla\mathbf{B}(\mathbf{b})|^2 = |T(\mathbf{b}) + \tfrac{1}{2}\mathbf{J}\times\mathbf{b}|^2 = |T(\mathbf{b})|^2 + \tfrac{1}{4}|\mathbf{J}\times\mathbf{b}|^2 + T(\mathbf{b})\cdot(\mathbf{J}\times\mathbf{b}). \tag{31}$$

We may cast (29) in a form emphasizing the symmetric role of S and T:

$$\begin{aligned}\frac{D\Psi}{Dt} &= -2(\mathbf{b}\cdot S(\mathbf{b}))^2 + |S(\mathbf{b})|^2 + \tfrac{1}{4}|\omega\times\mathbf{b}|^2 + S(\mathbf{b})\cdot(\omega\times\mathbf{b}) \\ &\quad + 2(\mathbf{b}\cdot T(\mathbf{b}))^2 - |T(\mathbf{b})|^2 - \tfrac{1}{4}|\mathbf{J}\times\mathbf{b}|^2 - T(\mathbf{b})\cdot(\mathbf{J}\times\mathbf{b}) \\ &\quad + F - \mathbf{b}\cdot P''_*(\mathbf{b}) + \mathbf{b}\cdot G(\mathbf{b}).\end{aligned} \tag{32}$$

This expression, however, is not very useful to estimate the contributions of the different terms. Let us instead start from the formulae in (26) and (27) to study the local effects of the field: since $\mathbf{B} = B\mathbf{b}$, $\mathrm{d}/\mathrm{d}s$ representing the differential along the arc length in the field line,

$$\mathbf{b}\cdot(\nabla\mathbf{B}) = \nabla\mathbf{B}(\mathbf{b}) = \frac{\mathrm{d}}{\mathrm{d}s}(B\mathbf{b}) = \frac{\mathrm{d}B}{\mathrm{d}s}\mathbf{b} + B\kappa\mathbf{n} \tag{33}$$

where $\mathbf{n}$ is the normal vector to the field line and κ is its curvature. Therefore

$$\begin{aligned}|\nabla\mathbf{B}(\mathbf{b})|^2 &= \left(\frac{\mathrm{d}B}{\mathrm{d}s}\right)^2 + B^2\kappa^2 \\ \mathbf{b}\cdot\nabla\mathbf{B}(\mathbf{b}) &= \frac{\mathrm{d}B}{\mathrm{d}s}.\end{aligned} \tag{34}$$

Therefore the whole second line of (32) may be substituted by

$$\left(\frac{\mathrm{d}B}{\mathrm{d}s}\right)^2 - B^2\kappa^2. \tag{35}$$

This term may in principle have either sign, but for chaotic flows it is far more likely to be negative. Since magnetic field lines are transported by the flow, they become rapidly extremely convoluted and their curvature is high in almost every point of them, whereas the variation of the field size along the line is not large. Even for geometries with large field gradients, such as the sheet or rope structures mentioned in the introduction, the largest variation is always transversal to the field line: the lines lie within the sheets and the field varies slowly along them. We may conclude that the contribution of the magnetic field gradient to the growth of Ψ is generally negative for the most interesting cases.

We have already discussed the difficulty of foreseeing the action of the nonlocal term $\mathbf{b} \cdot P_*''(\mathbf{b})$, which depends on the behaviour of velocity and field in the whole domain, not only in a neighbourhood of the point. The role of p_* is to balance the action of the kinetic and magnetic forces to conserve the volume, and we could say in a vague way that the contribution of this term is positive when the projection of these forces is concave along the magnetic field line, and negative if convex. Obviously, the effect of $\mathbf{b} \cdot G(\mathbf{b})$ depends on the forcing, and we have already proved that the mean action of F is zero.

We are left with the local kinetic term $-2(\mathbf{b} \cdot S(\mathbf{b}))^2 + |S(\mathbf{b})|^2 + \frac{1}{4}|\omega \times \mathbf{b}|^2 + S(\mathbf{b}) \cdot (\omega \times \mathbf{b})$, and we intend to study the efficiency with which it quenches magnetic field growth when this points in any eigendirection of S, such as the most unstable one. We always have $|\mathbf{b} \cdot S(\mathbf{b})| \leqslant |S(\mathbf{b})|$, but when $\mathbf{b}$ approaches any eigenvector of S with associated eigenvalue λ, both tend to the same value $|\lambda|$, and $S(\mathbf{b}) \cdot (\omega \times \mathbf{b})$ tends to zero. The term becomes

$$-\lambda^2 + \tfrac{1}{4}|\omega \times \mathbf{b}|^2 = -\Psi^2 + \tfrac{1}{4}|\omega \times \mathbf{b}|^2. \tag{36}$$

Thus the size of the eigenvalue has a negative influence on magnetic field growth: the larger it is, the more rapidly it tends to decrease. If we were allowed to omit all the remaining terms, the equation

$$\frac{D\Psi}{Dt} = -\Psi^2 \tag{37}$$

would yield a decrease of the order of t^{-1}, but this is too rough to be accurate. Leaving apart the rest, we have a positive contribution $(1/4)|\omega \times \mathbf{b}|^2$, which is associated with the growth of magnetic energy by rotation of the flow. Its effect is maximal when $\mathbf{b}$ is orthogonal to the rotation axis, and vanishes when it is collinear with it: there are otherwise no preferred directions. Since vorticity and magnetic field share many similarities (in fact they satisfy the same equation in the absence of Lorentz force), one could be tempted to conclude that they tend to become collinear and therefore the term tends to vanish. This view is reinforced by the fact that a classical dynamo mechanism generates a magnetic field in the direction of the rotation axis of the plasma flow, but it does not resist closer scrutiny. For one thing, when both velocity and field are planar and depend only on the two plane coordinates, field and vorticity are actually orthogonal all the time. No assumption about chaotic flow can help us: in the absence of forcing (or when this is a potential field), the total energy of the plasma is conserved, so that dynamo growth of the magnetic energy must be achieved at the expense of the kinetic one. Thus we have reason to believe

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 \, \mathrm{d}V < 0. \tag{38}$$

Since

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 \, \mathrm{d}V = -\int_\Omega \mathbf{u} \cdot \nabla \mathbf{u}(\mathbf{u}) \, \mathrm{d}V + \int_\Omega (\nabla \times (\mathbf{J} \times \mathbf{B})) \cdot \mathbf{u} \, \mathrm{d}V \tag{39}$$

and

$$\int_\Omega \mathbf{u} \cdot \nabla \mathbf{u}(\mathbf{u}) \, \mathrm{d}V = \frac{1}{2} \int_\Omega \mathbf{u} \cdot \nabla u^2 \, \mathrm{d}V = 0 \tag{40}$$

we are left with the Lorentz force term. Provided one of the magnitudes **u**, **B** or **J** vanishes at the boundary, this can be transformed:

$$\int_\Omega (\nabla \times (\mathbf{J} \times \mathbf{B})) \cdot \mathbf{u} \, \mathrm{d}V = \int_\Omega (\mathbf{J} \times \mathbf{B}) \cdot \omega \, \mathrm{d}V = -\int_\Omega (\omega \times \mathbf{B}) \cdot \mathbf{J} \, \mathrm{d}V. \tag{41}$$

Therefore the variation of the kinetic energy may be bounded as follows:

$$\left| \frac{1}{2} \int_\Omega u^2 \, \mathrm{d}V \right| \leqslant \left(\int_\Omega |\omega \times \mathbf{B}|^2 \, \mathrm{d}V \right)^{1/2} \left(\int_\Omega J^2 \, \mathrm{d}V \right)^{1/2}. \tag{42}$$

Obviously, $\omega \times \mathbf{b}$ cannot be too small in the mean for the dynamo to act.

4. Conclusions

One of the standard processes to obtain exponential growth of the magnetic energy in kinematic dynamos is to make the field point in the most unstable direction of the strain matrix; the second requirement is to minimize diffusive loss by positive folding. We ignore this second phenomenon by focusing on ideal plasmas, in order to study how this mechanism of magnetic field growth by stretching fares when considering the effect of the Lorentz force upon the flow, i.e. taking into account the whole MHD system. The main analysis tool is the equation satisfied by the rate of growth of magnetic energy, as well as the equation of the growth rate normalized to unit magnetic fields. The different terms affecting its evolution are the following: first, those imposed by the forcing, about which naturally everything depends on the nature of the independent term. Second, the action of the Hessian matrix of the total pressure on the magnetic field. Since the pressure is a nonlocal quantity depending on the global behaviour of velocity and magnetic field, this term depends upon the whole problem including boundary conditions and therefore nothing much can be said *a priori*, except that as a rule it cannot be neglected if we wish to avoid some absurd conclusions. Third is a fluctuating term of mean zero, and fourth is the effect of the torsion of the magnetic field along the local field line. This term, at least in chaotic flows, is most likely to be negative, thus decreasing the growth rate. Finally, the local kinetic effects due to the velocity gradient. It is shown that when the magnetic field lies near an eigenvector of the strain matrix, the square of the associated eigenvalue detracts from field growth, which means that the more unstable the direction where the field points, the more rapidly it tends to stabilize. There exists a positive contribution given by the local rotation of the plasma at the point, but this term does not have preferred directions of instability. The impression of the whole is that the flow tends to quench rapidly exponential field growths by making the magnetic field point away from unstable eigenvectors. This is in some contrast to the results of hydrodynamic turbulence, where it has been shown that the vorticity tends to align with the intermediate eigenvector of the strain matrix.

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